# End Semester Examination 

Quantum Mechanics II, M. Math.<br>September - December 2020.<br>Instructor: Prabuddha Chakraborty (pcphysics@gmail.com)<br>December $19^{\text {th }}, 2020$, Morning Session.<br>Duration: 4 hours.<br>Total points: 90.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. (a) Consider a spin- $\frac{1}{2}$ particle and let $\hat{S}_{i}, i=x, y, z$ be its spin-operators with the proper angular momentum commutation relations. Now, consider the eigenstate of $\hat{S}_{z}$ with the eigenvalue $\frac{\hbar}{2}$, with respect to which all following expectation values are taken.
i. Define the uncertainty of a given operator $\hat{O}$ as $\left\langle(\Delta \hat{O})^{2}\right\rangle=$ $\left\langle\hat{O}^{2}\right\rangle-\langle\hat{O}\rangle^{2}$. Using your result, check the generalized uncertainty relation

$$
\left\langle(\Delta \hat{A})^{2}\right\rangle\left\langle(\Delta \hat{B})^{2}\right\rangle \geq \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^{2}
$$

with $\hat{A} \rightarrow \hat{S_{x}}, \hat{B} \rightarrow \hat{S_{y}}$. [4]
ii. Check the same for the eigenstate of $\hat{S}_{x}$ with eigenvalue $\frac{\hbar}{2}$. [5]
(b) Consider the operator $\hat{R}=\frac{1}{\sqrt{2}}\left(1+\sigma_{x}\right)$. Prove that when it acts on a two-component normalized quantum state (a two-component "spinor") it acts as a matrix representation of the rotation operator causing a rotation about the $x$-axis by an angle $-\pi / 2$. [4]
(c) Suppose that a Hamiltonian of a system is a linear operator with the following conditions satisfied:

- $\hat{H}|\Psi\rangle=g|\Phi\rangle$
- $\hat{H}|\Phi\rangle=g^{*}|\Psi\rangle$
- $\hat{H}\left|\Upsilon_{n}\right\rangle=0$

Here $|\Phi\rangle$ and $|\Psi\rangle$ are a pair of arbitrary normalized, independent but not necessarily orthogonal state-vectors, $g$ is a complex constant, and $\left|\Upsilon_{n}\right\rangle$ runs over all state-vectors orthogonal to both $|\Phi\rangle$ and $|\Psi\rangle$.
i. What are the conditions satisfied by $|\Phi\rangle$ and $|\Psi\rangle$ such that the Hamiltonian is Hermitian? [3]
ii. If the conditions are satisfied, find the states with definite energy and their corresponding energy eigenvalues. [4]
2. (a) Consider a system with a Hamiltonian $\hat{H}$ and a pair of observable quantities $\hat{A}$ and $\hat{B}$. They follow the commutation relation relations with the Hamiltonian such as: $[\hat{H}, \hat{A}]=i \omega \hat{B}$ and $[\hat{H}, \hat{B}]=-i \omega \hat{A}$. The expectation values of $\hat{A}$ and $\hat{B}$ are known at $t=0$ in some state. Find expressions for the expectation values of $\hat{A}(t)$ and $\hat{B}(t)$. [5]
(b) Let us define the partition function of a system, $\mathbb{Z}$, as

$$
\mathbb{Z}=\left.\int_{\mathbb{V}^{\prime}} K\left(\overrightarrow{r^{\prime}}, t ; \overrightarrow{r^{\prime}}, 0\right)\right|_{\beta=i t / \hbar} d \mathbb{V}^{\prime}
$$

show that $\lim _{\beta \rightarrow \infty}-\frac{1}{\mathbb{Z}} \frac{\partial \mathbb{Z}}{\partial \beta}$ yields the ground state energy of the system. [5]
(c) Consider a three dimensional harmonic oscillator (i.e., it follows a potential that rises parabolically in all three Cartesian directions 1,2 , and 3 (you can think of them as $x, y$ and $z$ ). The potential is impossible to visualize in three dimensional universe, but it is perfectly possible to exist in it). The harmonic oscillator has mass $m$, and the following three frequencies $\omega_{1}<\omega_{2}<\omega_{3}$, for oscillations in the three different directions. The frequencies strictly satisfy the triangle inequality. Also assume that $\omega_{2}+\omega_{3}<2 \omega_{1}$.
i. Evaluate $\left\langle\hat{\mathbf{R}^{2}}(t)\right\rangle$ and $\left\langle\hat{\mathbf{P}^{2}}(t)\right\rangle$ for the ground state, second excited state (i.e., the state having the third lowest energy) and the fourth eigenstate of the quantum problem. Here $\mathbf{R}$ is the three-dimensional vector position operator and $\mathbf{P}$ is the threedimensional vector momentum operator. Feel free to choose your boundary condition(s) at $t=0$, but remember to explicitly specify them. $[4+4+4=12]$
ii. Demonstrate explicitly that the uncertainty principle is satisfied for the ground-state for all times. [3]
3. A system consists of two spin- $\frac{1}{2}$ particles. Observer A specializes in the measurements of the spin-components of particle 1 , and observer B specializes in the measurements of spin-components of particle 2. It is known that the system is in a spin-singlet state, i.e., $S_{t o t}=0$.
(a) What is the probability of observer A to obtain a measurement-result of $s_{1, z}=\frac{\hbar}{2}$ when observer B makes no measurement? [4]
(b) If observer B makes no measurement, what is the probability of observer A to obtain $s_{1, x}=\frac{\hbar}{2}$ ? [3]
(c) Now consider that the particle 1 on whom observer A does measurements was a spin-3/2 particle instead. Imagine Observer B has just done a measurement on $S_{2, z}$. Find the possible values of measurements and the probabilities of finding them for a $S_{1, x}$ measurement by A, if the result of B's experiment was $-\hbar / 2$ and the system was known to be in a $S_{t o t}=1$ state before all measurements. If there is not sufficient information to find the answer completely, explain why. [8]
(d) Imagine now a third observer C comes along who can only measure $S_{t o t}$ of the whole system. He puts the system in a Stern-Gerlach-type experiment and observes how many beams the system as a whole breaks into. Imagine that particle 2 is known to be a electron, but 1 can be anything. After conducting his expriment, C can only tell that particle 1 must be a boson (integer spin) of either spin- 1 or spin- 2 . How many beams did the resultant system beam break into? [5]
4. Consider a positronium (a bound state between an electron ( $\mathrm{e}^{-}$) and a positron $\left(\mathrm{e}^{+}\right)$). Both are spin- $1 / 2$ particles. Since they carry different charges, they are perfectly distinguishable particles. Let us call the spin operator of one of then $\overrightarrow{S_{1}}$ and the other's $\overrightarrow{S_{2}}$. Neglect the co-ordinate part of their wave-functions. The Hamiltonian governing their interactions in the presence of a uniform magnetic field B along the $z$-axis can be written as

$$
\hat{H}=A \overrightarrow{S_{1}} \cdot \vec{S}_{2}+\frac{e B}{m_{e} c}\left(S_{1 z}-S_{2 z}\right)
$$

(a) Considering the first term of the Hamiltonian much stronger than the second term in the sense of perturbation theory, solve the problem for all four energy levels up to second order in perturbation theory. The problem is time-independent; however, the use of non-degenerate or degenerate perturbation theory is left to your choice. [12]
(b) Now (i) solve for the four energy eigenvalues exactly analytically and (ii) compare the answers to the perturbation theory results you obtained in the first part. (iii) Show that the analytical result for the eigen-energies cannot be reproduced exactly in any finite order of perturbation theory for small (in the perturbation theory sense) but finite $B .[8+4+1=13]$

